Quantum Complexity: Entanglement of a Q-machine

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1 Introduction

1.1 Context

In the context of the Natural Computation and Self Organization class, ϵ-machines were widely discussed as classical objects used to simulate and study stochastic processes. Recently, a quantum analog of this object has been introduced [1]. The q-machine consists of a set $\{|\eta_k(L)\rangle\}$ of quantum states that are in one to one correspondence with the classical causal states $\sigma_k \in \mathcal{S}$. This quantum states are constructed as follows:

$$|\eta_j(L)\rangle = \sum_{w^L \in |\mathbf{A}|^L} \sum_{\sigma_k \in \mathbf{S}} \sqrt{Pr(w^L, \sigma_k|\sigma_j)}|w^L\rangle|\sigma_k\rangle$$

here, w^L denotes a length L word and $Pr(w^L, \sigma_k|\sigma_j)$ is the probability of observing word w^L and landing in state σ_k given that the system is in state σ_j. The Hilbert space of which this pure quantum states are elements is the product $\mathcal{H}_w \otimes \mathcal{H}_\sigma$. The first space corresponds to the space of words of length L, it is of size $|\mathcal{A}|^L$ where $|\mathcal{A}|$ is the alphabet size, it's basis elements are $|w^L\rangle$. The second space corresponds to the space of classical causal states, it's size is $|\mathcal{S}|$ and has basis elements $|\sigma_k\rangle$.

As can be seen from this definition of the states, the q-machine captures the statistics of the process: if a measurement is performed in the $\mathcal{H}_w$ space, a specific word w^L is measured with probability $Pr(w^L, \sigma_k|\sigma_j)$ and due to unifilarity the system is in state $|\sigma_k\rangle$, which corresponds to a classical causal state σ_k.

The initial state of the q-machine is defined then by:

$$\rho(L) = \sum_i \pi_i |\eta_i(L)\rangle\langle\eta_i(L)|$$

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with π is the stationary distribution of the classical causal states.

In this context, reference [1] introduces the quantum analog of the statistical complexity, C_q, which is defined as the Von Neumann entropy of ρ ($S(\rho) = Tr(\rho \log(\rho))$). Viewing the statistical complexity as the communication cost of synchronizing to a process, the q-machine is advantageous because there is strong evidence for claiming that $C_q \leq C_\mu$.

1.2 Motivation

The fact that the q-machine reduces the cost of synchronization for a process is a point in favor of the quantum representation, and it also incites looking for further advantages that this representation can contribute to the modelling of a stochastic process.

A natural quantity to start looking into is entanglement. This is a physical resource exclusive to quantum mechanical systems, and is responsible to many of the advantages of quantum information and quantum computation over its classical counterpart [5]. Despite the fact that entanglement has been studied for several decades now, a lot of aspects of it remain to be understood. The measurement of entanglement of a pure state of a bipartite system is well understood and standarized, but this is not the case for mixed states. In the case of mixed states there are several different measurements and they're used in different contexts. One of this measurements, which will be defined below is the entanglement of formation (EoF) of a mixed state of bipartite quantum states. This measurement was calculated for a simple example process (the Biased Coins) and the results are shown in Figure 1.

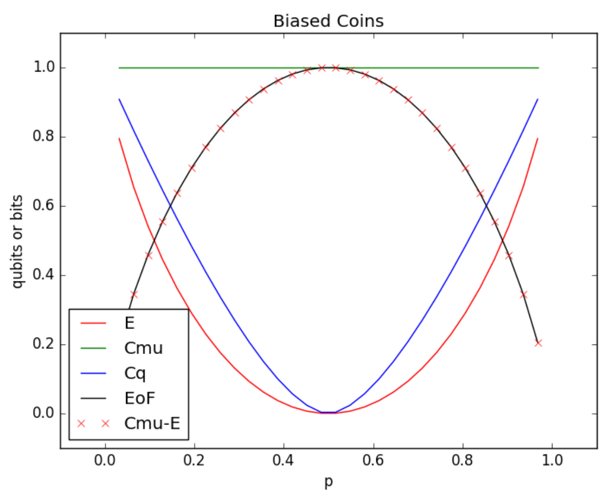

Figure 1: For the two biased coins process, the entanglement of formation is the same as the crypticity ($C_\mu - E$)

As can be seen from the graph, for this process we get $EoF = C_\mu - E$ which is a very surprising result because it suggests a relationship between a purely quantum quantity with classical quantities. The objective of this project was to explore this result further and try to understand what this relationship means. Even though this still remains to be answered, an advance towards understanding is presented as well as a set of more specific questions that might lead to a more clear picture of what the EoF of a q-machine says about the stochastic process.

2 Background

Since the Hilbert space that describes the q-machine states is defined as a bipartite space, we will only discuss entanglement in such quantum systems. As was previously mentioned there are several ways of measuring entanglement for mixed states, one of the most used measurements used in quantum information theory is entanglement of formation [5], which will be defined here. This measurement can be calculated analytically for mixed states of systems [2], but is yet to be defined for higher dimensional systems.

2.1 Entanglement of a pure state of two qubits

A pure quantum state of two qubits composed of spaces A and B, can be represented in the following way:

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \delta|10\rangle + \gamma|11\rangle$$

where $|\alpha|^2 + |\beta|^2 + |\delta|^2 + |\gamma|^2 = 1$ and the first qubit is from space A and the second one from space B. Such state is said to be entangled if it cannot be written as the product state of two qubits (i.e. quantum states of the form $|\Phi\rangle = a|0\rangle + b|1\rangle$ with $|a|^2 + |b|^2 = 1$) one from space A and the other from space B. The entanglement of state $|\Psi\rangle$ can then be measured as follows:

$$E(\Psi) = S(Tr_B|\Psi\rangle\langle\Psi|) = S(Tr_A|\Psi\rangle\langle\Psi|)$$

That is, one takes the partial trace of the density matrix $|\Psi\rangle\langle\Psi|$ (i.e. trace over the first or second qubit only). If the state is "separable" the resulting density matrix will be that of a pure state and its von Neumann entropy will be zero. Otherwise, it will have off-diagonal elements different than zero, so that the Von Neumann entropy of said matrix will be different than 0. This is the standard way of measuring entanglement for a pure system.

For example, take the separable state: $|\alpha\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$, we can calculate it's partial trace:

$$Tr_B|\alpha\rangle\langle\alpha| = |0\rangle\langle 0|$$

Now, the entanglement of α would be

$$E(\alpha) = S(|0\rangle\langle 0|) = 0$$

For a non-separable state, like $|B_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ we have:

$$|B_1\rangle\langle B_1| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |00\rangle\langle 00|)$$

So that

$$Tr_B(|B_1\rangle\langle B_1|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

and we get

$$E(B_1) = 1$$

This state is said to be maximally entangled and is one of the four Bell states:

$$|B_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|B_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|B_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|B_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

which are all in a sense "special" because they are maximally entangled and form an orthonormal set.

2.2 Entanglement of Formation

One way to interpret the entanglement of a pure state of two qubits, such as the ones mentioned above, is the entanglement of formation [2]. To do this, we take a particular entangled state as a standard unit of measure, for example the singlet state $|B_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Now, suppose that Alice and Bob (each holding one of the qubits) have a protocol that, given m shared of this $|B_4\rangle$ states, can create n "high fidelity" copies of $|\Psi\rangle$. This protocol can only involve local operations (each of them can only operate on their respective qubit) and they share a classical communication channel (this is called an "LOCC" protocol). The minimum value of the ration m/n that can be achieved in the limit of large n is the Entanglement of Formation, which turns out to be equal $E(\Psi)$ [4]. Schematically:

$$nE(\Psi) \times |B_4\rangle \to n \times |\Psi\rangle$$

So, for example if the value of $E(\Psi)$ is 1/2 one would need 500 singlet pairs to create 1000 copies of Φ. It is then reasonable to say that state Ψ has half of the entanglement of a singlet pair.

This idea can be extended to mixed states: Alice and Bob are now trying to create n copies of mixed state ρ. A naive approach to calculating the EoF for a mixed state is first writing ρ as:

$$\rho = \sum_j p_j |\Phi_j\rangle\langle\Phi_j|$$

where $|\Phi_j\rangle$ are distinct pure states of the bipartite systems and p_j are non-negative real numbers that add up to one. Now, one would expect [2] the EoF to be:

$$E_{average}(\rho) = \sum_j p_j E(\Phi_j)$$

But there is a problem with this definition, because it depends on the particular decomposition of ρ that was chosen. For example, if we have

$$\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

it can be built as an qual mixture of states $|00\rangle$ and $|11\rangle$ which would give an entanglement of 0, since this are pure states and no singlets are required to build ρ. But ρ can also be built as an equal mixture of $(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle))$ and $(\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle))$ which would yield an entanglement of 1. To solve this, one must concentrate on the idea that E is the minimum number of singlets required to create ρ, so one should consider the decomposition of ρ that minimizes $E_{average}$. The EoF of a mixed state ρ is then defined as:

$$E(\rho) = \inf \sum_j p_j E(\Phi_j)$$

Calculating that infimum is non-trivial in the general case, but there is a closed form for it for the case of a mixed state of two qubits [2]. In order to introduce said formula we must first introduce the notion of Concurrence.

2.3 Concurrence

Consider a pure state of two qubits $|\Phi\rangle$. The concurrence $C(\Phi)$ is defined as

$$C(\Phi) = |\langle\Phi|(\sigma_y \otimes \sigma_y)|\Phi^*\rangle|$$

where $|\Phi^*\rangle$ is the complex conjugate of $|\Phi\rangle$ and σ_y is the Pauli y matrix, which operating on a qubit sends it to the orthogonal state (diametrically opposite on the Bloch sphere). That is:

$$\sigma_y|0\rangle = i|1\rangle$$
$$\sigma_y|1\rangle = -i|1\rangle$$

The concurrence of a pure state of two qubits would then be zero if the state is separable. For any maximally entangled state on the other hand, the concurrence would be 1. So in it's own right the concurrence is a measurement of entanglement, but unlike the entanglement of formation it is not a resource-based or information-theoretic measure [2]. It has been proved that:

$$E(\Phi) = \mathcal{E}(C(\Phi))$$

where:

$$\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right)$$

and

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

2.4 EoF for a mixed state of two qubits

Now, for a mixed state ρ the concurrence is then defined as:

$$C(\rho) = \inf \sum_j p_j C(\Phi_j)$$

For two qubits, this can be calculated analitically [3] as:

$$C(\rho) = \max 0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4$$

here, the λ_is are the square roots of the eigenvalues of $\rho\tilde{\rho}$ in descending order. With:

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

The entanglement of formation of mixed state ρ can then be proved to be:

$$E(\rho) = \mathcal{E}(C(\rho))$$

This quantity has the same interpretation for mixed states as it does for pure states. It's the minimum number of mixed states of singlets needed to generate a state ρ.

3 Example Processes

Using the closed form of EoF for quantum mixed states of two qubits we can look at q-machines of two state processes. As was mentioned in the introduction, the first example studied yielded the interesting result that the EoF has the same form as a classical information quantity of the process. A more detailed description of this particular example and two other examples are studied in this section

3.1 Biased Coins Process

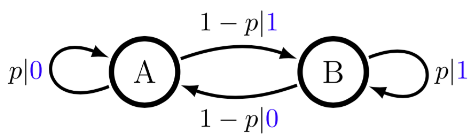

Figure 2: ϵ-machine for the Biased Coins process [1]

The epsilon machine of this process is shown in Figure 2. As can be seen this process has several symmetries. It is also Markov Order $R = 1$ and cryptic order $k = 1$, so it's a very "well behaved" process. From Figure 1 it can be seen that $EoF = C_\mu - E$. In Figure 3 we included the entropy rate of the process, which for this case is also equal to the EoF calculated for the process.

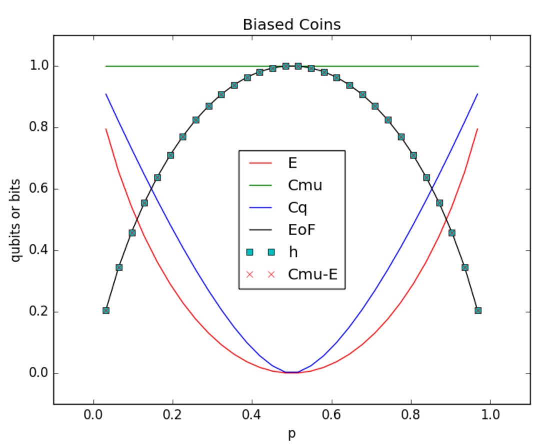

Figure 3: Information quantities for the Biased Coins Process. The graph shows that, in this case EoF is also equal to h_μ

3.2 Even Process

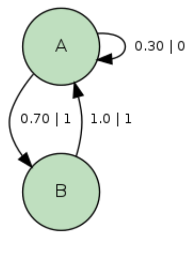

Figure 4: ϵ-machine for the Even Process with p = 0.3.

The Even Process's ϵ-machine is shown in Figure 4. The calculation of entanglement of formation is shown in Figure 5. For this example, it can be seen that EoF does not satisfy any of the equalities satisfied by the Biased Coins. It is worth noticing, however that h_μ seems to have a similar functional form. Also, exploring this example right after the Biased Coins Process might suggest that the changes seen in the graph are due to the fact that this process is Markov Order infinity and cryptic order zero; unfortunately the next example will show that the equalities are not satisfied in another process that has both R and k equal to one.

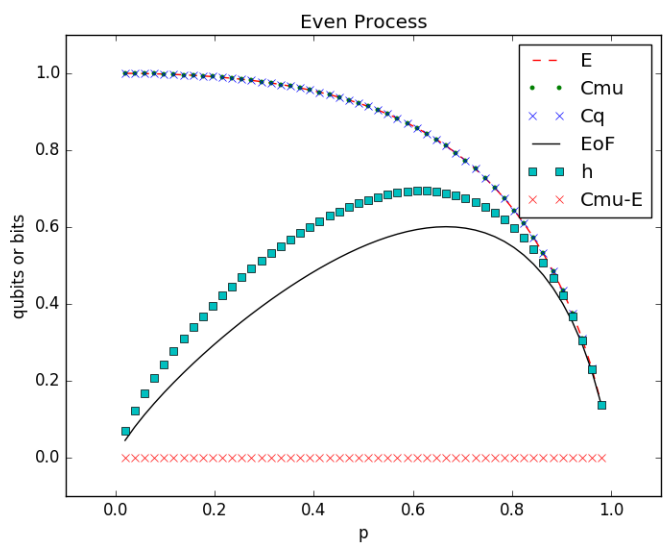

Figure 5: Classical information quantities and EoF for the Even Process

3.3 Golden Mean Process

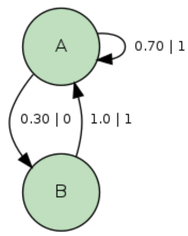

Figure 6: ϵ-machine for the Golden Mean Process with p = 0.3.

The Golden Mean Process's ϵ-machine is shown in Figure 6. Like the Biased Coins Process, the Golden Mean has R = 1 and k = 1, yet it does not satisfy the same equalities as the Biased Coins. We can see again that both EoF and h_μ have a similar functional form. It also seems reasonable to think that h_μ might be an upper bound for EoF, and this idea should be explored further.

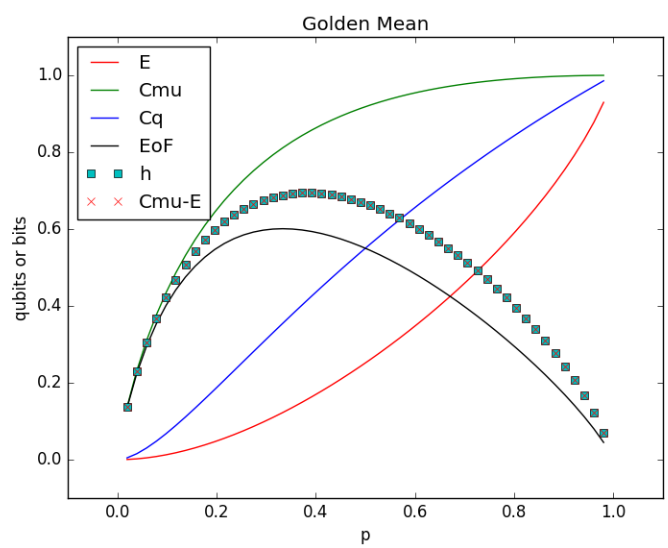

Figure 7: Classical information quantities and EoF for the Golden Mean Process.

3.4 Further comments on the examples

Even though Figure 2 suggests there is a relationship between the EoF and one or several of the classical information quantities of the process this is still not obvious and should not be taken lightly, since EoF is a quantum measurement. A couple of things are consistent among the three examples: both h_μ and EoF have similar functional forms, which makes sense considering the function $h(x)$ used to calculate the EoF has a similar form to the formula for h_μ of a unifilar process.

All the quantities in the graphs were obtained using CMPy functions and the QuTip library for Python. In order further understand the EoF we also calculated it "by hand" for the Biased Coins Process, and looked at the spin flipped density matrices $\tilde{\rho}$ for the other two processes. A potentially interesting result is that only for the case of the Biased Coins $\rho = \tilde{\rho}$, just as for a maximally entangled pure state the spin flipped state returns the original state. This should be considered further to see if it has any direct implications in the relationships of EoF shown in Figure 2.

4 Towards Interpretation

4.1 Entanglement of Formation as a measurement of Entanglement

The EoF as a measurement of entanglement has several advantages, two of which are that it can be calculated analytically for certain mixed states and that it is equal to the standard measurement of entanglement for the case of pure states, which is not true for all measurements of entangled. It has also been studied more than other measurements since it is relevant for quantum computational problems [5].

Despite all of this, there is still the problem of how to calculate it for bipartite systems composed of Hilbert spaces of size greater than two. Another potential disadvantage mentioned in [4] is that one would expect dimensionless quantities such as $E(\rho)/E(\sigma)$ to be independent of the units chosen to define E. For example, we defined here the standard unit as the singlet state, but in principle one can choose any entangled state. In general if one chooses a state other than a Bell state as a unit to measure E, the mentioned ratio is not the same as for when the Bell state is used.

One could disregard this problem by arguing that the Bell state is in a sense "special", or that choosing another mixed state as a unit of measurement is unnatural. This aren't strong enough arguments if one is truly determined to construct a robust measurement of entanglement, specially because in bipartite systems by definition of EoF we are implicitly assuming that all types of entanglement in this composite Hilbert space are the same and one can go from one entangled state to other. The ratio problem should not be enough to disqualify the EoF as a useful measurement, since it has a very clear interpretation and has also proved to be useful several contexts, such as quantum error-correction.

As an exploratory approach the EoF is a good measurement to use, but it would be interesting to eventually study the possibility of using some other measurements of entanglement. Also, the problem of interpreting the EoF in the context of q-machines remains open. Some remarks about that are explored in the next section.

4.2 Entanglement of Formation of a q-Machine

In the physical interpretation of EoF one assumes that an LOCC process can be implemented between the first and second qubit in order to transform from one entangled state to another, yet this does not seems to be the case for the q-machine states. First of all, it is not obvious who, what or how this local operations in the qubits would be performed. Also, the example processes that we are studying so far are described by bipartite states composed of two dimensional spaces, and even though both spaces behave mathematically the same they are not semantically the same, which should probably not be taken lightly.

Another thing is that so far we have been using the initial state of the machine, which is a mixed state, to calculate the EoF. Yet, if we consider the evolution of the process, the q-machine has to be collapsed and then rebuilt at every step, and once the process is in a known causal state, the new state of the q-machine is described by a pure state (one of the $|\eta_j\rangle$), so calculating the entanglement reduces to the problem of calculating the entanglement of a pure state. It would be interesting to look at how that entanglement changes for the different possible states of the machine.

The interpretation of what an EoF tells us about a process remains to be understood, let alone how to use it. Parallel to this, considering how to extend this calculation to higher dimensional spaces is still unclear.

4.3 Discussion

Besides the considerations already mentioned, the quantum mechanical states that are used in our description of stochastic processes have several constraints. Such constraints still need to be explored in the context of the more general expressions that can be found in the literature for EoF. This might not only bring some simplifications in the calculation (and thus a potentially reasonable way to calculate EoF for processes with more than two states), but also bring us closer to an interpretation of EoF that makes more sense as applied to a q-machine.

References

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